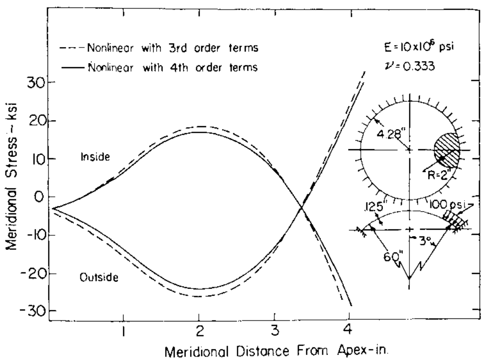

Fig. 4 Meridional stress along arc length.

difference becomes obvious when one compares the relative magnitudes of the strains and rotations. For example, at the point of maximum rotation of the shell, the midsurface strain $\hat{e}_s$ is approximately 0.0037 in./in. while the rotation $\hat{e}_{13}$ is 0.06 rad. Comparing the third-order term $\hat{e}_s\hat{e}_{13}^2$ to the fourth-order term $\hat{e}_{13}^4$, one finds that they contribute almost equally to the strain energy.

Figure 4 presents a comparison of the stresses obtained for a spherical cap under localized loading by the two nonlinear approximations. Again, it is observed that the fourth-order terms are significant, yielding an approximate 8% reduction in stresses when included in the analysis.

Figure 5 presents stress resultants for a shell with negative Gaussian curvature. The omission of the fourth-order terms results in an overestimation of the maximum inside stress by over 17% when compared to the solution obtained with the fourth-order terms. Again, it is noted that the largest deviation between the two approximations occurs in the apex region where the maximum rotations occur.

The importance of the fourth-order terms becomes quite evident in buckling analyses where the prebuckling deformations are large. For the problem in Fig. 5, retention of the fourth-order terms yields a symmetric buckling load of 290 psi whereas neglecting these terms yields a symmetric buckling load of 90 psi.

Based on the results presented here, it is concluded that the fourth-order terms in the strain energy expression previously neglected in Ref. 1 are indeed quite important and hence must be included in the large deflection shell analysis.

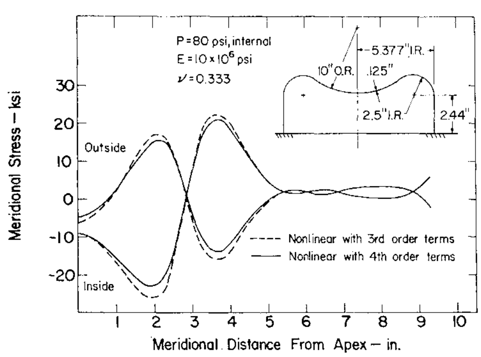

Fig. 5 Meridional stress along arc length.

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Time Variations of Generalized Spectral Functions for Density Turbulence

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Introduction

In a previous Note,[1] we illustrated the time variation of the over-all spectrum for a velocity fluctuation turbulence. The energy spectrum varied as k^4 as $k \to 0$. Here we look at the turbulence spectrum for a scalar field, such as the density or temperature fluctuation in a gas. For illustration, the scalar is specified to be electron density fluctuations in an ionized gas imbedded in a neutral background, although we neglect chemical effects and assume isotropic and homogeneous turbulence. For generality, we allow the spectrum to vary as some even power of k as $k \to 0$.

Electron Density Fluctuations

We denote the spectrum of electron density fluctuation by $S(k)$, normalized so that $\int S(k)k^2dk = 2\pi^2$. The energy spectrum is proportional to $k^2S(k)$, and $\langle(\delta n_e)^2\rangle$ is the mean square fluctuation in electron density. The generalized spectrum that we propose to use[2] as a model has the form

$$\langle(\delta n_e)^2\rangle S(k,t) = \langle(\delta n_e)^2\rangle r_0^3(2\pi)^{3/2}n! \frac{(kr_0)^{2n}[k^2r_0^2 + k_0^2r_0^2]^{(-\mu-2-2n)/4}}{2^{-n}(2n+1)!(k_0r_0)^{(-\mu+1)/2}} \frac{K_{\frac{1}{2}\mu+1+n}\{[k^2r_0^2 + k_0^2r_0^2]^{1/2}\}}{K_{(\mu-1)/2}(k_0r_0)} \quad (1)$$

where $S(k) \propto k^{-\mu-2}$ in an inertial range where $k_0 \ll k \ll r_0^{-1}$, and $S(k) \propto k^{2n}$ as k goes to zero, where n is a positive integer. We allow time variations in k_0, r_0 and in $\langle(\delta n_e)^2\rangle$, where k_0^{-1} and r_0 now refer to the scale lengths of the density fluctuation. The micro- and integral scale lengths are given by[2]

$$\lambda_{\delta n}^2 = 2(k_0r_0)^{(\mu-1)/2}K_{(\mu-1)/2}(k_0r_0)/[k_0^2(1+2n/3) \times (k_0r_0)^{(\mu-3)/2}K_{(\mu-3)/2}(k_0r_0)] \quad (2)$$

$$\Lambda_{\delta n} = \pi n!(k_0r_0)^{\mu/2}K_{\mu/2}(k_0r_0)/[2^{3/2}\Gamma(n+3/2) \times k_0(k_0r_0)^{(\mu-1)/2}K_{(\mu-1)/2}(k_0r_0)] \quad (3)$$

The Loitsianskii invariant based on this model is[2]

$$I_{\delta n} \equiv \lim_{k\to 0}\left[\frac{\langle(\delta n_e)^2\rangle S(k)}{2\pi^2k^{2n}}\right] = \langle(\delta n_e)^2\rangle\left(\frac{2}{\pi}\right)^{1/2} \times \frac{2^n n!}{(2n+1)!}\frac{(k_0r_0)^{(\mu/2)+1+n}K_{(\mu/2)+1+n}(k_0r_0)}{k_0^{2n+3}(k_0r_0)^{(\mu-1)/2}K_{(\mu-1)/2}(k_0r_0)} \quad (4)$$

The rate of dissipation of density fluctuations[3] $\epsilon_{\delta n}$ has dimensions $T^{-1}L^{-6}$ and is given by $\epsilon_{\delta n} \equiv -(d/dt)\langle(\delta n_e)^2\rangle = 12D_a\langle(\delta n_e)^2\rangle/\lambda_{\delta n}^2$, where D_a is the ambipolar diffusion coefficient. In this discussion, we let ν represent any combination of kinematic viscosity of the background gas and diffusion coefficient of the density, such as $\nu^{1-\alpha}D_a^{\alpha}$, having dimensions L^2/T.

We allow power-time dependences in the two limits of the initial and late decay periods, such that $\epsilon \propto t^{-\gamma}$, $\epsilon_{\delta n} \propto t^{-\xi}$, and $(k_0r_0) \propto (\epsilon t^2/\nu)^x$, where ϵ is the energy dissipation rate per unit mass of the background gas. The results for these

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Table 1 Power indices of time decay associated with density fluctuations if $\epsilon \propto t^{-\gamma}$, $\epsilon_{\delta n} \propto t^{-\xi}$, and $k_0 r_0 \propto (\epsilon t^2/\nu)^x \propto (t^{x(2-\gamma)})$

QUANTITY	SELF PRE-SERV-ING	INERTIAL RANGE REGION ($k_0 r_0 \ll 1$)							LATE DECAY REGION ($k_0 r_0 \gg 1$)	
		GENERAL		$\langle(\delta n_e)^2\rangle \propto k_0^{3+2n}$		$r_0 \propto \epsilon^{-1/4}$	$k_0 \propto \epsilon^{1/4}$	WEBB	GENERAL	$\langle(\delta n_e)^2\rangle \propto \left(\frac{k_0}{r_0}\right)^{3/2+n}$
		$1<\mu<3$	$\mu>3$	$1<\mu<3$	$\mu>3$	$1<\mu<3$	$\mu>3$	$\mu=\frac{5}{3}$		
$\epsilon_{\delta n}$	-4	$-\xi$	$-\xi$	$-\frac{5}{2}-n+(3+2n)(2-\gamma)(3-\mu)\frac{x}{2}$	$-\frac{5+2n}{2}$	$-\xi$	-4	$-\frac{7}{3}$	$-\xi$	$-\frac{5+2n}{2}$
ϵ	-2	$-\gamma$	$-\gamma$	$-\gamma$	$-\gamma$	$-\gamma$	-2	$-\frac{7}{3}$	$-\gamma$	$-\gamma$
$\langle(\delta n_e)^2\rangle$	-3	$1-\xi$	$1-\xi$	$-\frac{(3+2n)}{2}[1-(2-\gamma)(3-\mu)x]$	$-\frac{3+2n}{2}$	$1-\xi$	-3	$-\frac{4}{3}$	$1-\xi$	$-\frac{3+2n}{2}$
ϵt^2	const	$2-\gamma$	$2-\gamma$	$2-\gamma$	$2-\gamma$	$2-\gamma$	const	$-\frac{1}{3}$	$2-\gamma$	$2-\gamma$
k_0	$-\frac{1}{2}$	$\frac{-1+(2-\gamma)(3-\mu)x}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}+(2-\gamma)(3-\mu)\frac{x}{2}$	$-\frac{1}{2}$	$\frac{-4+\gamma(3-\mu)}{4(\mu-1)}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{2}+(2-\gamma)\frac{x}{2}$	$-\frac{1}{2}+(2-\gamma)\frac{x}{2}$
r_0	$\frac{1}{2}$	$\frac{1+(2-\gamma)(\mu-1)x}{2}$	$\frac{1+2x(2-\gamma)}{2}$	$\frac{1}{2}+(2-\gamma)(\mu-1)\frac{x}{2}$	$\frac{1}{2}+x(2-\gamma)$	$\frac{\gamma}{4}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{1}{2}+(2-\gamma)\frac{x}{2}$	$\frac{1}{2}+(2-\gamma)\frac{x}{2}$
$k_0 r_0$	const	$(2-\gamma)x$	$(2-\gamma)x$	$(2-\gamma)x$	$(2-\gamma)x$	$-\frac{(2-\gamma)}{2(\mu-1)}$	const	$\frac{1}{4}$	$x(2-\gamma)$	$x(2-\gamma)$
$\lambda_{\delta n}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\Lambda_{\delta n}$	$\frac{1}{2}$	$\frac{1-(2-\gamma)(3-\mu)x}{2}$	$\frac{1}{2}$	$\frac{1}{2}-(2-\gamma)(3-\mu)\frac{x}{2}$	$\frac{1}{2}$	$\frac{4-\gamma(3-\mu)}{4(\mu-1)}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$

two limits are shown in Table 1. The self-preserving solution arises with $\gamma = 2$, but it does not obey the Loitsianskii invariant. Another self-preserving solution that obeys the invariant (not shown in the table) is obtained by setting $x = 0$, and $\xi = (2n + 5)/2$. For $n = 0$, this corresponds to the solution given in Hinze (Ref. 3, p. 237). However, ϵt^2 is not constant in time. Other solutions are obtained by assuming the Loitsianskii invariant, viz., $\langle(\delta n_e)^2\rangle \propto k_0^{3+2n}$ for the initial stage and $\langle(\delta n_e)^2\rangle \propto (k_0/r_0)^{(3+2n)/2}$ for the final stage. Otherwise, we inspect all the various power law ranges in wave number for scalar additives postulated by various authors (see Ref. 3, pp. 232–4) and note that these can be deduced with the proportionalities that $r_0 \propto \epsilon^{-1/4}$ for $1 < \mu < 3$ and $k_0 \propto \epsilon^{1/4}$ for $\mu > 3$.

The entry in Table 1 under "WEBB" refers to his[4] considerations or those of Proudian and Feldman,[5] who show that far down an expanding wake, the integral scale length $\Lambda_{\delta n}$ varies with axial distance X as $X^{1/3}$. They also assume a direct proportionality between $\langle(\delta U)^2\rangle$ and $\langle(\delta n_e)^2\rangle$ and take $\mu = \frac{5}{3}$ (Kolmogorov's dependence). From the table in our previous note and from the present table under $r_0 \propto \epsilon^{-1/4}$, we deduce that $\xi = \lambda = \frac{7}{3}$, and then obtain the remaining results under "WEBB."

Another example is the work of Sutton.[6] He lets $\Lambda_g \propto \Lambda_{\delta n}$, that is, the same time decay for background velocity and density fluctuations as far as the integral scale length is concerned. Again using the table in our previous Note and the present table for the inertial range region satisfying the Loitsianskii invariant (with $n = 0$), we find $x = -1/(3 - \mu)$. This yields $\epsilon \propto t^{-\gamma}$, $\langle(\delta U)^2\rangle \propto t^{1-\gamma}$, $\Lambda_g \propto \Lambda_{\delta n} \propto k_0^{-1} \propto t^{(3-\gamma)/2}$, $\epsilon_{\delta n} \propto t^{(3\gamma-11)/2}$, $\langle(\delta n_e)^2\rangle \propto t^{3(\gamma-3)/2}$ and $\lambda_g \propto \lambda_{\delta n} \propto t^{1/2}$. These proportionalties are identical to those given by Sutton (with his $m = 3(3 - \gamma)/2$).

We have shown how time-dependent effects can be more naturally incorporated into a generalized spectrum function containing two scale lengths instead of a single scale length used in most of the previous approaches. Various dependences involving powers of time are deduced and compared with previous deductions, and these predictions are extended.

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Twisted Beam Element Matrices for Bending

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In this Note, the approximate theory of bending is employed to determine the element flexibility and stiffness matrices for a uniform beam element with linear geometric

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